Asymptotic expansions at any time for fractional scalar SDEs of Hurst index H>1/2

Sébastien Darses, Ivan Nourdin

Asymptotic expansions at any time for scalar fractional SDEs of Hurst index $H > 1/2$

Sébastien Darses and Ivan Nourdin

Université Pierre et Marie Curie
Laboratoire Probabilités et Modèles Aléatoires
Boîte courrier 188
75252 Paris Cedex 5, France
{sedarses,nourdin}@ccr.jussieu.fr

Abstract

We study the asymptotic expansions with respect to h of

$$E[\Delta_h f(X_t)], \quad E[\Delta_h f(X_t)|\mathscr{F}_t^X] \quad \text{and} \quad E[\Delta_h f(X_t)|X_t],$$

where $\Delta_h f(X_t) = f(X_{t+h}) - f(X_t)$, when $f : \mathbb{R} \to \mathbb{R}$ is a smooth real function, $t \geq 0$ is a fixed time, X is the solution of a one-dimensional stochastic differential equation driven by a fractional Brownian motion of Hurst index $H > 1/2$ and $\mathscr{F}^X$ is its natural filtration.

Key words: Asymptotic expansion - Fractional Brownian motion - stochastic differential equation - Malliavin calculus.

2000 Mathematics Subject Classification: Primary 60G15, 60G17; Secondary 60F15, 60H07.

1 Introduction

We study the asymptotic expansions with respect to h of

$$\begin{aligned} P_t f(h) &\triangleq E[\Delta_h f(X_t)], \\ \widehat{P}_t f(h) &\triangleq E[\Delta_h f(X_t)|\mathscr{F}_t^X] \\ \widetilde{P}_t f(h) &\triangleq E[\Delta_h f(X_t)|X_t], \end{aligned} \tag{1}$$

with $\Delta_h f(X_t) \triangleq f(X_{t+h}) - f(X_t)$, when $f : \mathbb{R} \to \mathbb{R}$ is a smooth real function, $t \geq 0$ is a fixed time, X is the solution to the following fractional stochastic differential equation:

$$X_t = x + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s) dB_s, \quad t \in [0, T], \tag{2}$$

and $\mathscr{F}^X$ is its natural filtration. Here, $b, \sigma : \mathbb{R} \to \mathbb{R}$ are real functions belonging to the space C_b^∞ of all bounded continuous functions having bounded derivatives of all order, while B is a one-dimensional fractional Brownian motion of Hurst index $H \in (1/2, 1)$. When the integral with respect to B is understood in the Young sense, Eq. (2) has a unique pathwise solution X in the set of processes whose paths are Hölder continuous of index $H \in (1 - H, H)$. Moreover, *e.g.* by Theorem 4.3 in [14], we have, for any $g : \mathbb{R} \to \mathbb{R} \in C_b^\infty$:

$$g(X_t) = g(x) + \int_0^t g'(X_s)\sigma(X_s) dB_s + \int_0^t g'(X_s) b(X_s) ds, \quad t \in [0, T]. \tag{3}$$

The asymptotic expansion of $E[f(X_h)]$ with respect to h has been recently studied in [1, 9]. It corresponds, in our framework, to the case where $t = 0$, since we have obviously

$$E[f(X_h) - f(x)] = P_0 f(h) = \widehat{P}_0 f(h) = \widetilde{P}_0 f(h).$$

In these latter references, the authors work in a multidimensional setting and under the weaker assumption that the Hurst index H of the fractional Brownian motion B is bigger than $1/3$ (the integral with respect to B is then understood in the rough paths sense of Lyons' type for [1] and of Gubinelli's type for [9]). In particular, it is proved in [1, 9] that there exists a family

$$\Gamma = \{\Gamma_{2kH+\ell} : \ (k, \ell) \in \mathbb{N}^2, \ (k, \ell) \neq (0, 0)\}$$

of differential operators such that, for any smooth $f : \mathbb{R} \to \mathbb{R}$, we have the following asymptotic expansion:

$$P_0 f(h) \underset{h \to 0}{\sim} \sum h^{2kH+\ell}\, \Gamma_{2kH+\ell}(f, \sigma, b)(x). \tag{4}$$

Moreover, in [9], operators $\Gamma_{2kH+\ell}$ are expressed using trees.

Now, a natural question arises. Can we also obtain a expansion of $P_t f(h)$ when $t \neq 0$? Let us first consider the case where B is the standard Brownian motion (it corresponds to the case where $H = 1/2$). By the Markov property, on one hand, we have $\widehat{P}_t f(h) = \widetilde{P}_t f(h)$ and, on the other hand, we always have $P_t f(h) = E\big[\widehat{P}_t f(h)\big]$. Thus, there exist relations between $P_t f(h)$, $\widehat{P}_t f(h)$ and $\widetilde{P}_t f(h)$. Moreover, the asymptotic expansion of $P_t f(h)$ can be obtained as a corollary of that of $P_0 f(h)$, by using the conditional expectation either with respect to the past $\mathscr{F}_t^X$ of X, or with respect to X_t alone, and the strong Markov property.

When $H > 1/2$, B is not Markovian. The situation concerning $P_t f(h)$, $\widehat{P}_t f(h)$ and $\widetilde{P}_t f(h)$ is then completely different and actually more complicated. In particular, we do not have $\widehat{P}_t f(h) = \widetilde{P}_t f(h)$ anymore and we cannot deduce the asymptotic expansion of $P_t f(h)$ from that of $P_0 f(h)$.

The current paper is concerned with the study of possible asymptotic expansions of the various quantities $P_t f(h)$, $\widehat{P}_t f(h)$ and $\widetilde{P}_t f(h)$ when $H > 1/2$. We will see that some non-trivial phenomena appear. More precisely, we will show in Section 3 that $\widehat{P}_t f(h)$ does not admit an asymptotic expansion in the scale of the fractional powers of h when $t \neq 0$. So, concerning $\widehat{P}_t f(h)$, the situations when $t = 0$ and $t > 0$ are really different. On the other hand, unlike $\widehat{P}_t f(h)$, the quantities $P_t f(h)$ and $\widetilde{P}_t f(h)$ admit, when $t \neq 0$, an asymptotic expansion in the scale of the fractional powers of h. However the computation of this expansion is more difficult than in the case where $t = 0$ (as made in [1, 9]). That is why we prefered only consider the one-dimensional case. Indeded, as an illustration, let us consider the trivial equation $dX_t = dB_t$, $t \in [0,T]$, $X_0 = 0$. That is $X_t = B_t$ for every $t \in [0,T]$. We have, by a Taylor expansion:

$$\widetilde{P}_0 f(h) = \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} E[(B_h)^k] + \ldots = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{f^{(2k)}(0)}{2^k k!} h^{2Hk} + \ldots,$$

while, by a linear Gaussian regression, when $t \neq 0$:

$$\begin{aligned} \widetilde{P}_t f(h) &= E\left[f\left((1 + \frac{H}{t} h - \frac{h^{2H}}{2t^{2H}} + \ldots) B_t + \left(h^{2H} + \ldots\right) N \right) - f(B_t) \middle| B_t \right] \\ &= \frac{H B_t f'(B_t)}{t} h - \frac{B_t f'(B_t)}{2t^{2H}} h^{2H} + \ldots, \end{aligned}$$

with $N \sim \mathscr{N}(0,1)$ a random variable independent of B_t.

One of the key point of our strategy relies on the use of a Girsanov transformation and the Malliavin calculus for fractional Brownian motion. We refer to [4, 12] for a deep insight of this topic.

We will restrict the exposition of our asymptotic expansions for the case when $\sigma = 1$. The reason is that, under the following assumption:

(A) The function σ is elliptic on $\mathbb{R}$, that is it verifies $\inf_{\mathbb{R}} |\sigma| > 0$, Eq. (2) can be reduced, using the change of variable formula (3), to a diffusion Y with a constant diffusion coefficient:

$$Y_t = \int_0^{X_t} \frac{dz}{\sigma(z)}.$$

Moreover, since $\int_0^{\cdot} \frac{dz}{\sigma(z)}$ is strictly monotonous from $\mathbb{R}$ to $\mathbb{R}$ under assumption (A), the σ-fields generated by X_t (resp. by X_s, $s \leq t$) and Y_t (resp. by Y_s, $s \leq t$) are the same. Consequently, to assume that $\sigma = 1$ is not at all restrictive since it allows to recover the general case under assumption (A). Consequently, we consider in the sequel that X is the unique solution of

$$X_t = x + \int_0^t b(X_s)\, ds + B_t, \quad t \in [0,T], \tag{5}$$

with $b \in \mathrm{C}_b^\infty$ and $x \in \mathbb{R}$.

The paper is organized as follows. In Section 2 we recall some basic facts about fractional Brownian motion, the Malliavin calculus and fractional stochastic differential equations. In Section 3 we prove that $\widehat{P}_t f(h)$ does not admit an asymptotic expansion with respect to the scale of fractional powers of h, up to order $n \in \mathbb{N}$. Finally, we show in Section 4 that $\widetilde{P}_t f(h)$ admits an asymptotic expansion.

2 Preliminaries

We begin by briefly recalling some basic facts about stochastic calculus with respect to a fractional Brownian motion. One refers to [11, 12] for further details. Let $B = (B_t)_{t\in[0,T]}$ be a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ defined on a probability space $(\Omega, \mathscr{A}, \mathbf{P})$. We mean that B is a centered Gaussian process with the covariance function $\mathrm{E}(B_s B_t) = R_H(s,t)$, where

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{6}$$

We denote by $\mathcal{E}$ the set of step $\mathbb{R}-$valued functions on [0,T]. Let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathfrak{H}} = R_H(t,s).$$

We denote by $|\cdot|_{\mathfrak{H}}$ the associate norm. The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to an isometry between $\mathfrak{H}$ and the Gaussian space $\mathcal{H}_1(B)$ associated with B. We denote this isometry by $\varphi \mapsto B(\varphi)$.

The covariance kernel $R_H(t,s)$ introduced in (6) can be written as

$$R_H(t,s) = \int_0^{s\wedge t} K_H(s,u) K_H(t,u) du,$$

where $K_H(t,s)$ is the square integrable kernel defined, for $s<t$, by

$$K_H(t,s) = c_H\, s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \tag{7}$$

with $c_H^2 = \frac{H(2H-1)}{\beta(2-2H,H-1/2)}$ and β the Beta function. By convention, we set $K_H(t,s)=0$ if $s \geq t$.

We define the operator $\mathcal{K}_H$ on $\mathrm{L}^2([0,T])$ by

$$(\mathcal{K}_H h)(t) = \int_0^t K_H(t,s) h(s) ds.$$

Let $\mathcal{K}_H^* : \mathcal{E} \to \mathrm{L}^2([0,T])$ be the linear operator defined by:

$$\mathcal{K}_H^*\left(\mathbf{1}_{[0,t]}\right) = K_H(t,\cdot).$$

The following equality holds for any $\phi,\psi \in \mathcal{E}$:

$$\langle \phi,\psi\rangle_{\mathfrak{H}} = \langle \mathcal{K}_H^*\phi, \mathcal{K}_H^*\psi\rangle_{\mathrm{L}^2([0,T])} = \mathrm{E}\left(B(\phi)B(\psi)\right).$$

Then, $\mathcal{K}_H^*$ provides an isometry between the Hilbert space $\mathfrak{H}$ and a closed subspace of $\mathrm{L}^2([0,T])$.

The process $W=(W_t)_{t\in[0,T]}$ defined by

$$W_t = B\big((\mathcal{K}_H^*)^{-1}(\mathbf{1}_{[0,t]})\big) \tag{8}$$

is a Wiener process, and the process B has the following integral representation:

$$B_t = \int_0^t K_H(t,s) dW_s.$$

Hence, for any $\phi \in \mathfrak{H}$,

$$B(\phi) = W\left(\mathcal{K}_H^*\phi\right).$$

If $b,\sigma \in \mathrm{C}_b^\infty$, then (2) admits a unique solution X in the set of processes whose paths are Hölder continuous of index $\alpha \in (1-H,H)$. Moreover, see *e.g.* [7], X has the following Doss-Sussman's type representation:

$$X_t = \phi(A_t,B_t), \quad t\in[0,T], \tag{9}$$

with ϕ and A given respectively by

$$\frac{\partial\phi}{\partial x_2}(x_1,x_2) = \sigma(\phi(x_1,x_2)), \quad \phi(x_1,0)=x_1, \qquad x_1,x_2\in\mathbb{R}$$

and

$$A_t' = \exp\left(-\int_0^{B_t} \sigma'(\phi(A_t,s))ds\right) b(\phi(A_t,B_t)), \quad A_0 = x_0, \qquad t\in[0,T].$$

Let $b \in \mathrm{C}_b^\infty$ and X be the solution of (5). Following [13], the fractional version of the Girsanov theorem applies and ensures that X is a fractional Brownian motion of Hurst parameter H, under the new probability $\mathbf{Q}$ defined by $d\mathbf{Q} = \eta^{-1} d\mathbf{P}$, where

$$\eta = \exp\left(\int_0^T \left(\mathcal{K}_H^{-1} \int_0^{\cdot} b(X_r)dr\right)(s)dW_s + \frac{1}{2}\int_0^T \left(\mathcal{K}_H^{-1} \int_0^{\cdot} b(X_r)dr\right)^2(s)ds \right). \tag{10}$$

Let $\mathscr{S}$ be the set of all smooth cylindrical random variables, *i.e.* of the form $F = f(B(\phi_1), \ldots, B(\phi_n))$ where $n \geqslant 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to B is the element of $\mathrm{L}^2(\Omega, \mathfrak{H})$ defined by

$$D_s^B F \;=\; \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\phi_1), \ldots, B(\phi_n))\phi_i(s), \quad s \in [0, T].$$

In particular $D_s^B B_t = \mathbf{1}_{[0,t]}(s)$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 \;=\; \mathrm{E}\left[F^2\right] + \mathrm{E}\left[|D.F|_{\mathfrak{H}}^2\right].$$

The Malliavin derivative D verifies the chain rule: if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is C_b^1 and if $(F_i)_{i=1,\ldots,n}$ is a sequence of elements of $\mathbb{D}^{1,2}$ then $\varphi(F_1, \ldots, F_n) \in \mathbb{D}^{1,2}$ and we have, for any $s \in [0, T]$:

$$D_s\, \varphi(F_1, \ldots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \ldots, F_n) D_s F_i.$$

The divergence operator δ is the adjoint of the derivative operator D. If a random variable $u \in \mathrm{L}^2(\Omega, \mathfrak{H})$ belongs to the domain of the divergence operator, that is if it verifies

$$|\mathrm{E}\langle DF, u\rangle_{\mathfrak{H}}| \le c_u \, \|F\|_{\mathrm{L}^2} \quad \text{for any } F \in \mathscr{S},$$

then $\delta(u)$ is defined by the duality relationship

$$\mathrm{E}(F\delta(u)) = \mathrm{E}\langle DF, u\rangle_{\mathfrak{H}},$$

for every $F \in \mathbb{D}^{1,2}$.

3 Study of the asymptotic expansion of $\widehat{P}_t f(h)$

Recall that $\widehat{P}_t f(h)$ is defined by (1), with X given by (5).

Definition 1 *We say that $\widehat{P}_t f(h)$ admits an asymptotic expansion with respect to the scale of fractional powers of h, up to order $n \in \mathbb{N}$, if there exist some real numbers $0 < \alpha_1 < \ldots < \alpha_n$ and some nontrivial random variables $C_1, \ldots, C_n \in \mathrm{L}^2(\Omega, \mathscr{F}_t^X)$ such that*

$$\widehat{P}_t f(h) = C_1 \, h^{\alpha_1} + \ldots + C_n \, h^{\alpha_n} + o(h^{\alpha_n}), \quad \text{as } h \to 0,$$

where $o(h^\alpha)$ stands for a random variable of the form $h^\alpha \phi_h$, with $E\big[\phi_h^2\big] \to 0$ as $h \to 0$.

If $\widehat{P}_t f(h)$ admits an asymptotic expansion in the sense of Definition 1 we must have, in particular, the existence of $\alpha > 0$ verifying

$$\lim_{h \to 0} h^{-\alpha} \widehat{P}_t f(h) \text{ exists in } \mathrm{L}^2(\Omega) \text{ and is not identically zero.}$$

But, we have:

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R} \in \mathrm{C}_b^\infty$ and $t \in (0, T]$. Assume moreover that*

$$\mathrm{Leb}\big(\{x \in \mathbb{R} : \, f'(x) = 0\}\big) = 0. \tag{11}$$

Then, as $h \to 0$, $h^{-\alpha} \widehat{P}_t f(h)$ converges in $\mathrm{L}^2(\Omega)$ if and only if $\alpha < H$. In this case, the limit is zero.

Remark 1 Since $\widehat{P}_0 f(h) = \widetilde{P}_0 f(h)$, we refer to Theorem 2 for the case where $t = 0$.

Proof. The proof is divided into two cases:

i) First case: $\alpha \in (0, 1]$. Since $H > 1/2$, let us first remark that $h^{-\alpha} \widehat{P}_t f(h)$ converges in $\mathrm{L}^2(\Omega)$ if and only if $h^{-\alpha} f'(X_t) E[X_{t+h} - X_t | \mathscr{F}_t^X]$ converges in $\mathrm{L}^2(\Omega)$. Indeed, we use a Taylor expansion:

$$|f(X_{t+h}) - f(X_t) - f'(X_t)(X_{t+h} - X_t)| \leq \frac{1}{2} \, |f''|_\infty |X_{t+h} - X_t|^2,$$

so that

$$|\widehat{P}_t f(h) - f'(X_t) E[X_{t+h} - X_t | \mathscr{F}_t^X]| \leq \frac{1}{2} \, |f''|_\infty E\big[|X_{t+h} - X_t|^2 | \mathscr{F}_t^X\big].$$

Thus, applying in particular Jensen formula:

$$\begin{aligned} & h^{-2\alpha} E\big[|\widehat{P}_t f(h) - f'(X_t) E[X_{t+h} - X_t | \mathscr{F}_t^X]|^2\big] \\ \leq \;& \frac{1}{4} \, |f''|_\infty^2 h^{-2\alpha} \, E\left[E\big[|X_{t+h} - X_t|^2 | \mathscr{F}_t^X\big]\right]^2 \\ \leq \;& \frac{1}{4} \, |f''|_\infty^2 h^{-2\alpha} \, E\left[|X_{t+h} - X_t|^4\right] = O(h^{4H - 2\alpha}). \end{aligned}$$

Since $\alpha \leq 1 < 2H$, we can conclude.

By (11) and the fact that X_t has a positive density on $\mathbb{R}$ (see *e.g.* [10], Theorem A), we have that $h^{-\alpha} f'(X_t) E[X_{t+h} - X_t | \mathscr{F}_t^X]$ converges in $\mathrm{L}^2(\Omega)$ if and only if $h^{-\alpha} E[X_{t+h} - X_t | \mathscr{F}_t^X]$ converges in $\mathrm{L}^2(\Omega)$.

For X given by (5), we have that $\mathscr{F}^X = \mathscr{F}^B$. Indeed, one inclusion is obvious, while the other one can be proved using (9). Moreover, since $b \in C_b^\infty$ and $\alpha < 1$, the term $h^{-\alpha} E[\int_t^{t+h} b(X_s) ds | \mathscr{F}_t^X]$ converges when $h \downarrow 0$. Therefore, we have due to (5) that $h^{-\alpha} E[X_{t+h} - X_t | \mathscr{F}_t^X]$ converges in $\mathrm{L}^2(\Omega)$ if and only if $h^{-\alpha} E[B_{t+h} - B_t | \mathscr{F}_t^B]$ converges in $\mathrm{L}^2(\Omega)$.

Set

$$Z_h^{(t)} = h^{-\alpha} E[B_{t+h} - B_t | \mathscr{F}_t^B] = h^{-\alpha} \int_0^t \big(K_H(t+h,s) - K_H(t,s)\big) dW_s,$$

where the kernel K_H is given by (7) and the Wiener process W is defined by (8). We have

$$\begin{aligned} \mathrm{Var}(Z_h^{(t)}) &= h^{-2\alpha} \int_0^t \big(K_H(t+h,s) - K_H(t,s)\big)^2 ds \\ &= h^{-2\alpha}\, c_H^2 \int_0^t s^{1-2H} \left(\int_t^{t+h} (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \right)^2 ds. \end{aligned}$$

We deduce

$$\begin{aligned} \mathrm{Var}(Z_h^{(t)}) &\geq h^{-2\alpha} \left(\frac{c_H}{H-\frac{1}{2}} \right)^2 t^{2H-1} \int_0^t s^{1-2H} \left((t+h-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}} \right)^2 ds \\ &= h^{-2\alpha} \left(\frac{c_H}{H-\frac{1}{2}} \right)^2 \int_0^t \left(1 - \frac{s}{t}\right)^{1-2H} \left((s+h)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds \\ &= h^{2(H-\alpha)} \left(\frac{c_H}{H-\frac{1}{2}} \right)^2 \int_0^{\frac{t}{h}} \left(1 - \frac{hs}{t}\right)^{1-2H} g^2(s) ds, \end{aligned} \tag{12}$$

with $g(s) = (s+1)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}$. Similarly,

$$\begin{aligned} \mathrm{Var}(Z_h^{(t)}) &\leq h^{-2\alpha} \left(\frac{c_H}{H-\frac{1}{2}} \right)^2 (t+h)^{2H-1} \int_0^t s^{1-2H} \left((t+h-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}} \right)^2 ds \\ &= h^{-2\alpha} \left(1 + \frac{h}{t}\right)^{2H-1} \left(\frac{c_H}{H-\frac{1}{2}} \right)^2 \int_0^t \left(1 - \frac{s}{t}\right)^{1-2H} \left((s+h)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds \\ &= h^{2(H-\alpha)} \left(1 + \frac{h}{t}\right)^{2H-1} \left(\frac{c_H}{H-\frac{1}{2}} \right)^2 \int_0^{\frac{t}{h}} \left(1 - \frac{hs}{t}\right)^{1-2H} g^2(s) ds. \end{aligned} \tag{13}$$

Note that $g^2(s) \sim (H - \frac{1}{2})^2 s^{2H-3}$ as $s \to +\infty$. So $s g^2(s) \longrightarrow 0$, as $s \to +\infty$, and $\int_0^{+\infty} |g^2(s)| ds < +\infty$ since $2H - 3 < -1$. Since $s \mapsto s g^2(s)$ is bounded on $\mathbb{R}^+$, we have by dominated convergence theorem that

$$\int_0^{\frac{t}{h}} \left(\left(1 - \frac{hs}{t}\right)^{1-2H} - 1 \right) g^2(s)ds = \int_0^1 \frac{(1-u)^{1-2H} - 1}{u} g^2\left(\frac{tu}{h}\right) \frac{tu}{h} du$$

tends to zero as $h \to 0$. Thus:

$$\lim_{h\to 0} \int_0^{\frac{t}{h}} \left(1 - \frac{hs}{t}\right)^{1-2H} g^2(s)ds = \int_0^\infty g^2(s)ds < +\infty.$$

Now, combined with (12)-(13), we deduce that

$$\mathrm{Var}(Z_h^{(t)}) \sim h^{2(H-\alpha)} \left(\frac{c_H}{H - \frac{1}{2}}\right)^2 \int_0^\infty g^2(s)ds, \quad \text{as } h \to 0. \tag{14}$$

If $Z_h^{(t)}$ converges in $\mathrm{L}^2(\Omega)$ as $h \to 0$, then $\lim_{h\to 0} \mathrm{Var}(Z_h^{(t)})$ exists and is finite. But, thanks to (14), we have that $\lim_{h\to 0} \mathrm{Var}(Z_h^{(t)}) = +\infty$ when $\alpha > H$. Consequently, $Z_h^{(t)}$ does not converge in $\mathrm{L}^2(\Omega)$ as $h \to 0$ when $\alpha > H$.

Conversely, when $\alpha < H$, we have from (14) that $\lim_{h\to 0} \mathrm{Var}(Z_h^{(t)}) = 0$. Then $Z_h^{(t)} \xrightarrow{\mathrm{L}^2} 0$ when $\alpha < H$.

In order to complete the proof of the first case, it remains to consider the case where $\alpha = H$. We first deduce from (14) that $Z_h^{(t)} \xrightarrow{\text{Law}} \mathscr{N}(0, \sigma_H^2)$, as $h \to 0$, with

$$\sigma_H^2 = \left(\frac{c_H}{H - \frac{1}{2}}\right)^2 \int_0^\infty g^2(s)ds.$$

Let us finally show that the previous limit does not hold in L^2. Assume for a moment that $Z_h^{(t)}$ converges in $\mathrm{L}^2(\Omega)$ as $h \to 0$. Then, in particular, $\{Z_h^{(t)}\}_{h>0}$ is Cauchy in $\mathrm{L}^2(\Omega)$. So, by denoting $Z^{(t)}$ the limit in $\mathrm{L}^2(\Omega)$, we have $E[Z_\varepsilon^{(t)} Z_\delta^{(t)}] \to E[|Z^{(t)}|^2]$ when $\varepsilon, \delta \to 0$. But, for any fixed $x > 0$, we can show, by using exactly the same transformations as above: as $h \to 0$,

$$E(Z_{hx}^{(t)} Z_{\frac{h}{x}}^{(t)}) \longrightarrow \left(\frac{c_H}{H - \frac{1}{2}}\right)^2 r(x) = E(|Z^{(t)}|^2),$$

where

$$\begin{aligned} r(x) &= \int_0^\infty \left((s+x)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right) \left((s + \frac{1}{x})^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right) ds \\ &= x \int_0^\infty g(x^2 u) g(u) du. \end{aligned}$$

Consequently, the function r is constant on $]0,+\infty[$. The Cauchy-Schwarz inequality yields:

$$|g|_{L^2}^2 = r(1) = r(\sqrt{2}) = \langle \sqrt{2}\, g(2\,\cdot), g\rangle_{L^2} \leqslant \sqrt{2}|g(2\,\cdot)|_{L^2}|g|_{L^2} = |g|_{L^2}^2.$$

Thus, we have equality in the previous inequality. We deduce that there exists $\lambda \in \mathbb{R}$ such that $g(2u) = \lambda g(u)$ for all $u \geq 0$. Since $g(0) = 1$ we have $\lambda = 1$. Consequently, for any $u \geq 0$ and any integer n, we obtain

$$g(u) = g\left(\frac{u}{2^n}\right) \xrightarrow[n\to\infty]{} g(0) = 1,$$

which is absurd. Therefore, when $\alpha = H$, $Z_h^{(t)}$ does not converge in $\mathrm{L}^2(\Omega)$ as $h \to 0$, which concludes the proof of the first case.

ii) Second case: $\alpha \in (1,+\infty)$. If $h^{-\alpha}\widehat{P}_t f(h)$ converges in $\mathrm{L}^2(\Omega)$, then $h^{-1}\widehat{P}_t f(h)$ converges in $\mathrm{L}^2(\Omega)$ towards zero. This contradicts the first case, which concludes the proof of Theorem 1. □

4 Study of the asymptotic expansion of $\widetilde{P}_t f(h)$

Recall that $\widetilde{P}_t f(h)$ is defined by (1), where X is given by (5). The main result of this section is the first point of the following theorem:

Theorem 2 *Let $t \in [0,T]$ and $f : \mathbb{R} \to \mathbb{R} \in \mathrm{C}_b^\infty$. We write $\mathcal{N}$ for $\mathbb{N}^2 \setminus \{(0,0)\}$. For $(p,q) \in \mathcal{N}$, set*

$$J_{2pH+q} = \{(m,n) \in \mathcal{N} :\ 2mH + n \leq 2pH + q\}.$$

1. *If $t \neq 0$, there exists a family $\{Z_{2mH+n}^{(t)}\}_{(m,n)\in\mathcal{N}}$ of random variables measurable with respect to X_t such that, for any $(p,q) \in \mathcal{N}$:*

$$\widetilde{P}_t f(h) = \sum_{(m,n)\in J_{2pH+q}} Z_{2mH+n}^{(t)}\, h^{2mH+n} + o(h^{2pH+q}). \tag{15}$$

2. *If $t = 0$, for any $(p,q) \in \mathcal{N}$, we have:*

$$\begin{aligned} P_0 f(h) &= \widetilde{P}_0 f(h) = \widehat{P}_0 f(h) \\ &= \sum_{(m,n)\in J_{2pH+q}} \left(\sum_{I\in\{0,1\}^{2m+n},\,|I|=2m} c_I\, \Gamma_I(f,b)(x) \right) h^{2mH+n} \\ &\qquad + o(h^{2pH+q}), \end{aligned}$$

with c_I and Γ_I respectively defined by (17) and (19) below.

Remark 2 1. In (15), $Z^{(t)}_{0H+1}$ coincides with the stochastic derivative of X with respect to its present t, as defined in [2].

2. The expansion (15) allows to obtain the expansion of $P_t f(h)$ for $t \neq 0$:

$$P_t f(h) = E\big[\widetilde{P}_t f(h)\big] = \sum_{(m,n)\in J_{2pH+q}} E\big[Z^{(t)}_{2mH+n}\big]\, h^{2mH+n} + o(h^{2pH+q}).$$

The following subsections are devoted to the proof of Theorem 2. Note that a quicker proof of the first assertion could be the following: once $t > 0$ is fixed, we write

$$X_{t+h} = X_t + \int_t^{t+h} b(X_s)ds + \widetilde{B}^{(t)}_h, \quad h \geq 0, \tag{16}$$

where $\widetilde{B}^{(t)}_h = B_{t+h} - B_t$ is again a fractional Brownian motion. Then, we could think that an expansion for $\widetilde{P}_t f(h)$ follows directly from the one for $\widetilde{P}_0 f(h)$, simply by a shift. It is unfortunately not the case, due to the fact that the initial value in (16) is not just a *real number* as in the case $t = 0$, but a *random variable*. Consequently, the computation of $E[\widetilde{B}^{(t)}_h | X_t]$ is not trivial, since $\widetilde{B}^{(t)}_h$ and X_t are not independent.

4.1 Proof of Theorem 2, point (2).

The proof of this part is in fact a direct consequence of Theorem 2.4 in [9]. But, for the sake of completeness on one hand and taking into account that we are dealing with the one-dimensional case on the other hand, we give all the details here. Indeed, contrary to the multidimensional case, it is easy to compute explicitly the appearing coefficients (see Lemma 1 below and compare with Theorem 31 in [1] or Proposition 5.4 in [9]) which has, from our point of view, also its own interest.

The differential operators Γ_I appearing in Theorem 2 are recursively* defined by

$$\Gamma_{(0)}(f,b) = bf', \qquad \Gamma_{(1)}(f,b) = f',$$

and, for $I \in \{0,1\}^k$,

$$\Gamma_{(I,0)}(f,b) = b\big(\Gamma_I(f,b)\big)', \qquad \Gamma_{(I,1)}(f,b) = \big(\Gamma_I(f,b)\big)', \tag{17}$$

with $(I,0),(I,1) \in \{0,1\}^{k+1}$. The constants c_I are explained as follows. Set

$$dB^{(i)}_t = \begin{cases} dB_t & \text{if} \quad i = 1, \\ dt & \text{if} \quad i = 0. \end{cases} \tag{18}$$

*We can also use a rooted trees approach in order to define the Γ_I's. See [9] for a thorough study, even in the multidimensional case and $H > 1/3$.

Then, for a sequence

$$I = (i_1, \ldots, i_k) \in \{0,1\}^k,$$

we define

$$c_I = E\left[\int_{\Delta^k[0,1]} dB^I\right] = E\left[\int_0^1 dB_{t_k}^{(i_k)} \int_0^{t_k} dB_{t_{k-1}}^{(i_{k-1})} \cdots \int_0^{t_2} dB_{t_1}^{(i_1)}\right]. \quad (19)$$

Set $|I| = \sum_{1 \leqslant j \leqslant k} i_j$. Equivalently, $|I|$ denotes the number of integrals with respect to 'dB'. Note, since B and $-B$ have the same law, that we have

$$c_I = c_I\,(-1)^{|I|}.$$

Thus c_I vanishes when $|I|$ is odd. In general, the computation of the coefficients c_I can be made as follows:

Lemma 1 *Let $I \in \{0,1\}^k$. We denote by $J = \{j_1 < \ldots < j_m\}$ the set of indices $j \in \{1, \ldots, k\}$ such that $dB_t^{(j)} = dt$. We then have that c_I is given by*

$$\int_0^1 dt_{j_m} \cdots \int_0^{t_{j_2}} dt_{j_1} E\left[\frac{\left(B_1 - B_{t_{j_m}}\right)^{k-j_m} \left(B_{t_{j_1}}\right)^{j_1-1}}{(k-j_m)!(j_1-1)!} \prod_{k=2}^{m} \frac{\left(B_{t_{j_k}} - B_{t_{j_{k-1}}}\right)^{j_k-j_{k-1}}}{(j_k-j_{k-1})!}\right].$$

The expectation appearing in the above formula can always be computed using the moment generating function of an m-dimensional Gaussian random variable. For instance, we have

$$\begin{aligned}
E\left[\int_0^1\int_0^{t_3}\int_0^{t_2} dt_1\, dB_{t_2}\, dB_{t_3}\right] &= \frac{1}{2(2H+1)} \qquad (20)\\
E\left[\int_0^1\int_0^{t_3}\int_0^{t_2} dB_{t_1}\, dt_2\, dB_{t_3}\right] &= \frac{2H-1}{2(2H+1)}\\
E\left[\int_0^1\int_0^{t_3}\int_0^{t_2} dB_{t_1}\, dB_{t_2}\, dt_3\right] &= \frac{1}{2(2H+1)}\\
E\left[\int_0^1\int_0^{t_4}\int_0^{t_3}\int_0^{t_2} dt_1\, dt_2\, dB_{t_3}\, dB_{t_4}\right] &= \frac{1}{2(2H+1)(2H+2)}\\
E\left[\int_0^1\int_0^{t_4}\int_0^{t_3}\int_0^{t_2} dt_1\, dB_{t_2}\, dt_3\, dB_{t_4}\right] &= \frac{H}{(2H+1)(2H+2)}\\
E\left[\int_0^1\int_0^{t_4}\int_0^{t_3}\int_0^{t_2} dt_1\, dB_{t_2}\, dB_{t_3}\, dt_4\right] &= \frac{1}{2(2H+1)(2H+2)}\\
E\left[\int_0^1\int_0^{t_4}\int_0^{t_3}\int_0^{t_2} dB_{t_1}\, dt_2\, dB_{t_3}\, dt_4\right] &= \frac{H}{(2H+1)(2H+2)}\\
E\left[\int_0^1\int_0^{t_4}\int_0^{t_3}\int_0^{t_2} dB_{t_1}\, dt_2\, dt_3\, dBt_4\right] &= \frac{H(2H-1)}{2(2H+1)(2H+2)}\\
E\left[\int_0^1\int_0^{t_4}\int_0^{t_3}\int_0^{t_2} dB_{t_1}\, dB_{t_2}\, dt_3\, dt_4\right] &= \frac{1}{2(2H+1)(2H+2)}.
\end{aligned}$$

Lemma 2 *When $f : \mathbb{R} \to \mathbb{R} \in \mathrm{C}_b^\infty$, we have*

$$\begin{aligned} f(X_h) &= f(x) + \sum_{k=1}^{n-1} \sum_{I_k \in \{0,1\}^k} \Gamma_{I_k}(f,b)(x) \int_{\Delta^k[0,h]} dB^{I_k}(t_1, \ldots, t_k) \\ &\quad + \sum_{I_n \in \{0,1\}^n} \int_{\Delta^n[0,h]} \Gamma_{I_n}(f,b)(X_{t_1})\, dB^{I_n}(t_1, \ldots, t_n), \quad (21) \end{aligned}$$

where, again using the convention (18), for $g : \mathbb{R} \to \mathbb{R} \in \mathrm{C}_b^\infty$,

$$\int_{\Delta^k[0,h]} g(X_{t_1})\, dB^{I_k}(t_1, \ldots, t_k) \triangleq \int_0^h dB_{t_k}^{(i_k)} \int_0^{t_k} dB_{t_{k-1}}^{(i_{k-1})} \cdots \int_0^{t_2} dB_{t_1}^{(i_1)}\, g(X_{t_1}).$$

Proof. Applying (3) already twice, we can write

$$\begin{aligned} f(X_h) &= f(x) + \int_0^h f'(X_s) dB_s + \int_0^h (bf')(X_s) ds \\ &= f(x) + \Gamma_{(1)}(f,b)(x)\, B_h + \Gamma_{(0)}(f,b)(x) h \\ &\quad + \sum_{I_2 \in \{0,1\}^2} \int_{\Delta^2[0,h]} \Gamma_{I_2}(f,b)(X_{t_1})\, dB^{I_2}(t_1, t_2). \end{aligned}$$

Applying (3) repeatedly we finally obtain (21). □

The remainder can be bounded by the following lemma:

Lemma 3 *If $n \geq 2$, $\varepsilon > 0$ (small enough) and $g : \mathbb{R} \to \mathbb{R} \in \mathrm{C}_b^\infty$ are fixed, we have,*

$$\sum_{I_n \in \{0,1\}^n} E \left| \int_{\Delta^n[0,h]} g(X_{t_1})\, dB^{I_n}(t_1, \ldots, t_n) \right| = O(h^{nH-\varepsilon}).$$

Proof. It is a direct application of Theorem 2.2 in [8], combined with the Garsia, Rodemich and Rumsey Lemma [6]. □

Thus, in order to obtain the asymptotic expansion of $\widetilde{P}_t f(h)$, Lemmas 2 and 3 say that it is sufficient to compute

$$E \left[\int_{\Delta^k[0,h]} dB^{I_k}(t_1, \ldots, t_k) \right],$$

for any $I_k \in \{0,1\}^k$, with $1 \leq k \leq n-1$. By the self-similarity and the stationarity of fractional Brownian motion, we have that

$$\int_{\Delta^k[0,h]} dB^{I_k}(t_1, \ldots, t_k) \stackrel{\mathcal{L}}{=} h^{H|I_k| + k - |I_k|} \int_{\Delta^k[0,1]} dB^{I_k}(t_1, \ldots, t_k).$$

Hence it follows

$$\mathrm{E}\left[\int_{\Delta^k[0,h]} dB^{I_k}(t_1,\ldots,t_k)\right] = h^{H|I_k|+k-|I_k|}c_{I_k}$$

and the proof of point (2) of Theorem 2 is a consequence of Lemmas 2 and 3 above.

4.2 Proof of Theorem 2, point (1).

Let $f:\mathbb{R}\to\mathbb{R}\in \mathrm{C}_b^\infty$ and X be the solution of (5). We then know (see Section 2) that X is a fractional Brownian motion of Hurst index H, under the new probability $\mathbf{Q}$ defined by $d\mathbf{Q}=\eta^{-1}d\mathbf{P}$, with η given by (10). Since $b:\mathbb{R}\to\mathbb{R}\in \mathrm{C}_b^\infty$, remark that $\eta\in\mathbb{D}^{1,2}$. Moreover, the following well-known formula holds for any $\xi\in\mathrm{L}^2(\mathbf{P})\cap\mathrm{L}^2(\mathbf{Q})$:

$$E\big[\xi|X_t\big] = \frac{E^{\mathbf{Q}}\big[\eta\xi|X_t\big]}{E^{\mathbf{Q}}\big[\eta|X_t\big]}.$$

In particular,

$$E\big[f(X_{t+h})-f(X_t)|X_t\big] = \frac{E^{\mathbf{Q}}\big[\eta\big(f(X_{t+h})-f(X_t)\big)|X_t\big]}{E^{\mathbf{Q}}\big[\eta|X_t\big]}.$$

Now, we need the following technical lemma:

Lemma 4 *Let $\zeta\in\mathbb{D}^{1,2}(\mathfrak{H})$ be a random variable. Then, for any $h>0$, the conditional expectation $E\big[\zeta(f(B_{t+h})-f(B_t))|B_t\big]$ is equal to*

$$\begin{aligned}
&H(2H-1)f'(B_t)\int_0^T du\, E[D_u\zeta|B_t]\int_t^{t+h}|v-u|^{2H-2}dv\\
&+\frac{1}{2}t^{-2H}f'(B_t)\big(B_tE\left[\zeta|B_t\right]-E\big[\langle D\zeta,\mathbf{1}_{[0,t]}\rangle_{\mathfrak{H}}|B_t\big]\big)\big(h^{2H}-(t+h)^{2H}+t^{2H}\big)\\
&-\frac{H}{2}t^{-2H}f''(B_t)E\big[\zeta|B_t\big]\int_t^{t+h}\left((v-t)^{2H-1}-v^{2H-1}\right)\left(t^{2H}+v^{2H}-(v-t)^{2H}\right)dv\\
&+\frac{1}{2}f''(B_t)E\big[\zeta|B_t\big]\big((t+h)^{2H}-t^{2H}\big)\\
&+H(2H-1)\int_0^T du\int_t^{t+h}|v-u|^{2H-2}E[D_u\zeta\big(f'(B_v)-f'(B_t)\big)|B_t]dv\\
&+Ht^{-2H}B_t\int_t^{t+h}\left((v-t)^{2H-1}-v^{2H-1}\right)E\left[\zeta\big(f'(B_v)-f'(B_t)\big)|B_t\right]dv\\
&-Ht^{-2H}\int_t^{t+h}\left((v-t)^{2H-1}-v^{2H-1}\right)E\big[\langle D\zeta,\mathbf{1}_{[0,t]}\rangle_{\mathfrak{H}}\big(f'(B_v)-f'(B_t)\big)|B_t\big]dv
\end{aligned}$$

$$-\frac{H}{2}t^{-2H}\int_t^{t+h}\left((v-t)^{2H-1}-v^{2H-1}\right)\left(t^{2H}+v^{2H}-(v-t)^{2H}\right)E\big[\zeta\big(f''(B_v)-f''(B_t)\big)|B_t\big]dv$$
$$+H\int_t^{t+h}E\left[\zeta\big(f''(B_v)-f''(B_t)\big)|B_t\right]v^{2H-1}dv. \tag{22}$$

Proof. Let $g:\mathbb{R}\to\mathbb{R}\in\mathrm{C}_b^1$. We can write using the Itô formula (5.44) p.294 in [11] and basics identities of Malliavin calculus:

$$\begin{aligned}
& E\left[\zeta g(B_t)(f(B_{t+h})-f(B_t))\right]\\
= \;& E\left[\zeta g(B_t)\delta(f'(B_\cdot)\mathbf{1}_{[t,t+h]})\right]+H\int_t^{t+h}E\left[\zeta g(B_t)f''(B_v)\right]v^{2H-1}dv\\
= \;& E\left[g(B_t)\langle D\zeta,\mathbf{1}_{[t,t+h]}f'(B_\cdot)\rangle_{\mathfrak{H}}\right]+E\big[\zeta g'(B_t)\langle\mathbf{1}_{[0,t]},\mathbf{1}_{[t,t+h]}f'(B_\cdot)\rangle_{\mathfrak{H}}\big]\\
& +H\int_t^{t+h}E\left[\zeta g(B_t)f''(B_v)\right]v^{2H-1}dv\\
= \;& H(2H-1)\int_0^T du\int_t^{t+h}|v-u|^{2H-2}E[f'(B_v)\,g(B_t)\,D_u\zeta]dv\\
& +H(2H-1)\int_0^t du\int_t^{t+h}(v-u)^{2H-2}E[\zeta\,f'(B_v)\,g'(B_t)]dv\\
& +H\int_t^{t+h}E\left[\zeta g(B_t)f''(B_v)\right]v^{2H-1}dv.
\end{aligned}$$

But

$$\begin{aligned}
E\left[\zeta g(B_t)f'(B_v)B_t\right] &= E\left[\langle D(\zeta g(B_t)f'(B_v)),\mathbf{1}_{[0,t]}\rangle_{\mathfrak{H}}\right]\\
&= E\big[g(B_t)f'(B_v)\langle D\zeta,\mathbf{1}_{[0,t]}\rangle_{\mathfrak{H}}\big]+E\big[\zeta g'(B_t)f'(B_v)\big]t^{2H}\\
&\quad+\frac{1}{2}E\big[\zeta g(B_t)f''(B_v)\big](t^{2H}+v^{2H}-(v-t)^{2H}).
\end{aligned}$$

Consequently

$$\begin{aligned}
& E\left[\zeta g(B_t)(f(B_{t+h})-f(B_t))\right]\\
= \;& H(2H-1)\int_0^T du\int_t^{t+h}|v-u|^{2H-2}E[f'(B_v)\,g(B_t)\,D_u\zeta]dv\\
& +H(2H-1)t^{-2H}\int_0^t du\int_t^{t+h}(v-u)^{2H-2}E\left[\zeta g(B_t)f'(B_v)B_t\right]dv\\
& -H(2H-1)t^{-2H}\int_0^t du\int_t^{t+h}(v-u)^{2H-2}E\big[g(B_t)f'(B_v)\langle D\zeta,\mathbf{1}_{[0,t]}\rangle_{\mathfrak{H}}\big]dv\\
& -\frac{1}{2}H(2H-1)t^{-2H}\int_0^t du\int_t^{t+h}(v-u)^{2H-2}E\big[\zeta g(B_t)f''(B_v)\big](t^{2H}+v^{2H}-(v-t)^{2H})dv\\
& +H\int_t^{t+h}E\left[\zeta g(B_t)f''(B_v)\right]v^{2H-1}dv.
\end{aligned}$$

We deduce:

$$
\begin{aligned}
& E\left[\zeta(f(B_{t+h})-f(B_t))|B_t\right] \\
= \quad & H(2H-1)\int_0^T du\int_t^{t+h}|v-u|^{2H-2}E[f'(B_v)D_u\zeta|B_t]dv \\
& +Ht^{-2H}B_t\int_t^{t+h}\left((v-t)^{2H-1}-v^{2H-1}\right)E\left[\zeta f'(B_v)|B_t\right]dv \\
& -Ht^{-2H}\int_t^{t+h}\left((v-t)^{2H-1}-v^{2H-1}\right)E\left[f'(B_v)\langle D\zeta,\mathbf{1}_{[0,t]}\rangle_{\mathfrak{H}}|B_t\right]dv \\
& -\frac{H}{2}t^{-2H}\int_t^{t+h}\left((v-t)^{2H-1}-v^{2H-1}\right)\left(t^{2H}+v^{2H}-(v-t)^{2H}\right)E\left[\zeta f''(B_v)|B_t\right]dv \\
& +H\int_t^{t+h}E\left[\zeta f''(B_v)|B_t\right]v^{2H-1}dv.
\end{aligned}
$$

Finally, (22) follows. □

First, apply the previous lemma with $\zeta=\eta$, $E=E^{\mathbf{Q}}$ and $B=X$, with η given by (10), $d\mathbf{Q}=\eta^{-1}d\mathbf{P}$ and X given by (5). Remark that $\eta\in\mathbb{D}^{\infty,2}$ (see e.g. Lemma 6.3.1 in [11], and [2] for the expression of Malliavin derivatives via the transfer principle). In particular, we can deduce that each random variable V_k, recursively defined by $V_0=\eta$ and $V_{k+1}=\langle DV_k,\mathbf{1}_{[0,t]}\rangle_{\mathfrak{H}}$ for $k\geqslant 0$, belongs to $\mathbb{D}^{1,2}$.

In (22), the deterministic terms $\int_t^{t+h}|v-u|^{2H-2}dv$, $\quad(t+h)^{2H}-t^{2H}$ and

$$\int_t^{t+h}\left((v-t)^{2H-1}-v^{2H-1}\right)\left(t^{2H}+v^{2H}-(v-t)^{2H}\right)dv$$

have a Taylor expansion in h of the type (15). Lemma 4 allows to obtain the first term of the asymptotic expansion using that $\int_t^{t+h}\phi(s)ds=h\phi(t)+o(h)$ for any continuous function ϕ. By a recursive argument using again Lemma 4, we finally deduce that (15) holds, which concludes the proof of Theorem 2.

Acknowledgments. The computations (20) have been made by A. Neuenkirch. We would like to thank him.

References

[1] F. Baudoin and L. Coutin (2007): *Operators associated with a stochastic differential equation driven by fractional Brownian motions.* Stoch. Proc. Appl. **117** (5), 550-574.

[2] S. Darses and I. Nourdin (2007): *Stochastic derivatives for fractional diffusions.* Ann. Probab., to appear.

[3] S. Darses, I. Nourdin and G. Peccati (2007): *Differentiating σ-fields analyzed into Gaussian settings.* Prepublication Paris VI.

[4] L. Decreusefond and A.S. Üstünel (1999): *Stochastic analysis of the fractional Brownian motion.* Potential Anal. **10**, 177-214.

[5] H. Föllmer (1984): *Time reversal on Wiener space.* Stochastic processes - mathematics and physics (Bielefeld). Lecture Notes in Math. **1158**, 119-129.

[6] A. Garsia, E. Rodemich and H. Rumsey (1970): *A real variable lemma and the continuity of paths of some Gaussian processes.* Indiana Univ. Math. Journal **20**, 565-578.

[7] F. Klingenhöfer and M. Zähle (1999): *Ordinary differential equations with fractal noise.* Proc. Amer. Math. Soc. **127** (4), 1021-1028.

[8] T. Lyons (1994): *Differential equations driven by rough signals. I. An extension of an inequality of L.C.Young.* Math. Res. Lett. **1**, no. 4, 451-464.

[9] A. Neuenkirch, I. Nourdin, A. Rößler and S. Tindel (2006): *Trees and asymptotic expansions for fractional diffusion processes.* Prepublication Paris VI and Darmstadt.

[10] I. Nourdin and T. Simon (2006): *On the absolute continuity of one-dimensional SDEs driven by a fractional Brownian motion.* Statist. Probab. Lett. **76**(9), 907-912.

[11] D. Nualart (2006): The Malliavin Calculus and Related Topics. Springer Verlag. Second edition.

[12] D. Nualart (2003): *Stochastic calculus with respect to the fractional Brownian motion and applications.* Contemp. Math. **336**, 3-39.

[13] D. Nualart and Y. Ouknine (2002): *Regularization of differential equations by fractional noise.* Stoch. Proc. Appl. **102**, 103-116.

[14] M. Zähle (1998): *Integration with respect to fractal functions and stochastic calculus I.* Probab. Th. Relat. Fields **111**, 333-374.